Large mapping tori of free group endomorphisms

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Abstract

We present an algorithm which, given any finite presentation of a group as input, will terminate with answer yes if and only if the group is large. We use this to prove that a mapping torus of a finitely generated free group automorphism is large if it contains a $\mathbb{Z} \times \mathbb{Z}$ subgroup of infinite index. We then extend this result to mapping tori of finitely generated free group endomorphisms, as well as showing that such a group is large if it contains a Baumslag-Solitar group of infinite index and has a finite index subgroup with first Betti number at least 2. We also show that if a group possesses a deficiency 1 presentation where one of the relators is a commutator then it is $\mathbb{Z} \times \mathbb{Z}$, large or is as far as possible from being residually finite.

1 Introduction

Recall that a finitely generated group G is large if it has a finite index subgroup possessing a homomorphism onto a non-abelian free group. The advantage of this notion, as explained in [41] and [22], is that on defining a group theoretic property to be a large property if it is preserved by preimages, finite index subgroups and supergroups, and quotienting out by finite normal subgroups, then G being large implies that G possesses any large

property $\mathcal{P}$, provided only that there is some finitely generated group with $\mathcal{P}$. Examples of such properties which are relevant to us include G containing a non-abelian free subgroup, being SQ-universal (every countable group is a subgroup of a quotient of G), having finite index subgroups with arbitrarily large first Betti number, having exponential word growth, and having subgroup growth of strict type n^n, which is the largest possible growth for finitely generated groups (see [34] Section 1.11 for definitions and proof that this is a large property). Thus on proving that G is large we obtain all these other properties for free.

There have been a range of results that give criteria for finitely generated or finitely presented groups to be large. Starting with B. Baumslag and S. J. Pride [2] which showed that groups with a presentation of deficiency at least 2 are large, we then have in [25] a condition that implies this result, as well as a proof that a group with a deficiency 1 presentation in which one of the relators is a proper power is large. This latter result was also independently derived by Stöhr in [42] and was followed by conditions for a group with a deficiency 0 presentation where some of the relators are proper powers to be large, due to Edjvet in [21]. Then further conditions for a finitely presented group to be large, all of which imply the Baumslag-Pride result, are by Howie in [27], the New York Group Theory Cooperative in [37] Chapter IV Theorem 7 and a characterisation by Lackenby in [30]. However in Section 2 we produce an algorithm which takes as input any finite presentation and we prove that it will terminate with answer yes if and only if the group G given by that presentation is large. If G is not large then the algorithm will not terminate in general unless G is finite. The algorithm operates by converting one of the relevant criteria into a statement about the Alexander polynomial of G. This can be checked by direct calculation and then we show that G being large means that this statement is true for a finite index subgroup of G.

We do not address running times and only briefly consider the practicalities of implementing the algorithm, although its simplicity should mean that it is of use. Instead we establish the effectiveness of the algorithm by using it to prove that a substantial class of finitely presented groups not previously known to be large has this property. Our focus in this paper is on groups of deficiency 1 as here the algorithm takes a suitably nice form. Of course unlike groups of deficiency 2 or higher, not all groups of deficiency 1 are large: think of $\mathbb{Z}$ or the soluble Baumslag-Solitar groups given by the presentations $\langle x, y | xyx^{-1} = y^m \rangle$ for $m \in \mathbb{Z} \backslash \{0\}$. Other examples of non-large deficiency

1 groups were given by Pride, Edjvet and Howie in [22] consisting of those Baumslag-Solitar groups $\langle x, y | xy^l x^{-1} = y^m \rangle$ for $l, m \neq 0$ where l and m are coprime, as well some HNN extensions of these, and one can find the odd further example in the literature.

As for large groups of deficiency 1, we have already mentioned those with a relator that is a proper power and we again have examples in [22] with Theorem 6 stating that the group $\langle x, y | x^n y^l x^{-n} = y^m \rangle$ for $l, m, n \neq 0$ is large if $|n| > 1$ or if l and m are not coprime. Further results of a more technical nature which give largeness for some other 2-generator 1-relator presentations are in [20] from 1984. Here it is asked if those groups which are an extension of a finitely generated non-abelian free group by $\mathbb{Z}$ are large. They are certainly torsion free groups with a natural deficiency 1 presentation and are also referred to as mapping tori of finitely generated non-abelian free group automorphisms. Some ad hoc examples of this type were shown there to be large. More recently a naturally occurring class of deficiency 1 groups was shown to be large in [15], [11], [31], namely fundamental groups of compact orientable irreducible 3-manifolds with (non-empty) boundary consisting solely of tori (with $\langle x, y | xyx^{-1} = y^m \rangle$ for $m = 0, \pm 1$ making up the few small exceptions).

At this point it seems difficult to say convincingly either way whether groups of deficiency 1 are generally large. In this paper we hope to offer substantial evidence that largeness is a natural property to expect in a deficiency 1 group. Although we will display a few new groups of deficiency 1 which are not large in Section 5, our main results are on establishing families of deficiency 1 groups which are all large. In Section 3 we prove that a mapping torus G of a finitely generated non-abelian free group automorphism is large if it contains a $\mathbb{Z} \times \mathbb{Z}$ subgroup. By [4], [5] and [8], these are exactly the mapping tori of finitely generated non-abelian free group automorphisms which are not word-hyperbolic, thus this question is now reduced to one only involving word-hyperbolic groups. However we make no use here of the notion of word-hyperbolicity: it is the $\mathbb{Z} \times \mathbb{Z}$ subgroup itself which is crucial to the proof of largeness. On combination with other results the method also deduces that if G is finitely generated but is F-by-$\mathbb{Z}$ for F an infinitely generated free group then G is large.

Mapping tori of finitely generated free group automorphisms appear to make up a sizeable class of deficiency 1 groups but we can expand this class considerably by allowing arbitrary endomorphisms in place of automorphisms. Such groups have been the attention of much recent research

where significant progress has been made. In particular these groups have been shown to be coherent (every finitely generated subgroup is finitely presented) in [23], Hopfian in [24] and even residually finite in [6]. If largeness were added to this list (on removing the obvious small exceptions) then it would show that such a mapping torus, indeed even a group which is virtually such a mapping torus, has all the nice properties that one could reasonably hope for (but not more than this, for instance G is never LERF if the endomorphism is injective but not surjective so this seems like much too strong a property to expect in an arbitrary deficiency 1 group).

In Section 4 we indicate how our proof of largeness for automorphisms generalises to mapping tori of finitely generated free group endomorphisms with a $\mathbb{Z} \times \mathbb{Z}$ subgroup, unless G is isomorphic to $\langle x, y | xyx^{-1} = y^{\pm 1} \rangle$. Unlike the case for automorphisms, this does not cover all such G which are not word-hyperbolic because on allowing endomorphisms we can now have G containing other soluble Baumslag-Solitar groups. It is conjectured in [28] that a mapping torus of a finitely generated free group endomorphism is word-hyperbolic if and only if it does not contain Baumslag-Solitar subgroups. We also obtain in Section 4 largeness for mapping tori G of finitely generated free group endomorphisms which contain a Baumslag-Solitar subgroup provided G has a finite index subgroup H ($\neq \mathbb{Z} \times \mathbb{Z}$) with $\beta_1(H) \geq 2$. Of course if $\beta_1(H) = 1$ for all H then we would have an example of such a G which is not large. However we know of no examples apart from the soluble Baumslag-Solitar groups themselves, and it seems believable that no other G has this property, in which case on assumption of this and the conjecture above we have largeness for all non word-hyperbolic mapping tori of finitely generated free group endomorphisms apart from the soluble Baumslag-Solitar groups above. But once again the actual concept of word-hyperbolicity is not used anywhere.

The proof for largeness of mapping tori of finitely generated free group endomorphisms containing $\mathbb{Z} \times \mathbb{Z}$ uses the fact that they are residually finite. However the condition actually required in the proof is much weaker and this is explored in Section 5. We introduce the concept of a residually useless group and this has a number of equivalent definitions, one of which is that it is finitely generated and non-abelian but has no non-abelian finite quotients. It is merely the property of not being residually useless which allows us to finish off the proof of largeness of our mapping tori. Moreover introducing this definition provides us with two advantages, one of which is theoretical and one of which is practical: first we get to weaken significantly our hypotheses

for largeness and second it should be much quicker on being given a particular presentation to determine that it is not residually useless by finding one non-abelian finite quotient rather than having to prove it is residually finite. We obtain Theorem 5.5 which states that if G has a deficiency 1 presentation in which one of its relators is a commutator then $G = \mathbb{Z} \times \mathbb{Z}$ or G is residually useless with abelianisation $\mathbb{Z} \times \mathbb{Z}$ or G is large. In particular, on excluding the obvious small group, the only way a group G with such a presentation can fail to be large is when it is as far away from being residually finite as G possibly can. This can ultimately be regarded as our main result in the sense that our proof of largeness for mapping tori involves a series of steps showing that they have finite index subgroups with a presentation of this form. The theorem also gives us largeness for groups G with a 2-generator 1-relator presentation where the relator is a commutator, unless $G = \mathbb{Z} \times \mathbb{Z}$ (which is easily detected) or G is residually useless. It is true that 2-generator 1-relator groups which are residually useless exist, but if we insist that the relator is a product of commutators then no examples are known; indeed it was only recently that non residually finite examples of such groups were given. Moreover if the relator is a single commutator then no examples are known that fail even to be residually finite, so in this case being not residually useless and hence large seems like a good bet.

2 The algorithm

We first need to summarise the facts we require about the Alexander polynomial and the Fox derivatives of a finitely presented group; see [16]. Let G be given by a finite presentation $\langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ and let G' be the derived (commutator) subgroup of G, so that the abelianisation $\overline{G} = G/G'$ is a finitely generated abelian group $\mathbb{Z}^b \times T$, where T is the torsion and $\mathbb{Z}^b$ is what we call the free abelianisation $ab(G)$ of G. Here $b = \beta_1(G)$ is the first Betti number of G and we must have $b \leq n$. The Alexander polynomial Δ_G of G is an element of the group ring $\mathbb{Z}[ab(G)]$ which we think of as Laurent polynomials in b variables with integer coefficients, but it is only specified up to units which are the monomials. It is defined by the following process: given the free group F_n of rank n with free basis $x_1, \ldots, x_n$ we have the free derivations (or Fox derivatives) $D_j : \mathbb{Z}[F_n] \to \mathbb{Z}[F_n]$ such that $D_j(x_i) = \delta_{ij}$. We construct the $m \times n$ Alexander matrix (a_{ij}) of our presentation for G by calculating $D_j(r_i) \in \mathbb{Z}[F_n]$ and then the entry a_{ij} is the image of $D_j(r_i)$

under the natural map from $\mathbb{Z}[F_n]$ to $\mathbb{Z}[ab(G)]$ via $\mathbb{Z}[G]$. We then define the Alexander ideal of the presentation to be the ideal of $\mathbb{Z}[ab(G)]$ generated by the $(n-1)\times(n-1)$ minors (namely those obtained by deleting one column and the appropriate number of rows) and the Alexander polynomial is the highest common factor of these minors, up to units. The utility of Δ_G is that it is independent of the finite presentation for G (at least once a basis is chosen for $ab(G)$ and we can certainly cope with a basis change), which can be seen because it is invariant under Tietze transformations.

If $\beta_1(G) = 0$ then Δ_G is just a non-zero integer, and probably not an interesting one at that, but our focus here is when $\beta_1(G) \geq 2$, in which case we require the definition of the relative Alexander polynomial. Let $f : G \to ab(G)$ be the free abelianisation map and let θ be any group homomorphism of G onto a free abelian group $\mathbb{Z}^k$, so that $k \leq b$ and $\theta = \tilde{\theta} f$. Then the Alexander polynomial $\Delta_{G,\theta}$ relative to θ is formed in exactly the same way with θ in place of f, and we can see that evaluation of Δ_G under $\tilde{\theta}$ divides $\Delta_{G,\theta}$. The important case for us is the polynomial $\Delta_{G,\chi}(t)$ when $\chi : G \to \mathbb{Z}$ is a surjective homomorphism, as then $\Delta_{G,\chi}$ carries specific and accessible information about ker χ. We quote the following essential fact.

Proposition 2.1 *Suppose G is a finitely presented group and $\chi : G \to \mathbb{Z}$ is a surjective homomorphism with ker $\chi = K$. Then the relative Alexander polynomial $\Delta_{G,\chi} \in \mathbb{Z}[t^{\pm 1}]$ has degree $\beta_1(K)$, with $\Delta_{G,\chi} = 0$ if and only if $\beta_1(K)$ is infinite.*

Proof. See [32]. Here we interpret $\beta_1(K) = \infty$ as ker χ surjects to $\mathbb{Z}^k$ for all $k \in \mathbb{N}$.

□

This immediately gives us restrictions on $\beta_1(K)$.

Proposition 2.2 *If G is finitely presented with $\beta_1(G) \geq 2$ and $K =$ ker χ for χ a surjective homomorphism of G to $\mathbb{Z}$ then the degree of $\Delta_{G,\chi}$ is at least $\beta_1(G) - 1$. However if $\beta_1(G) = 1$ then $\Delta_G(1) \neq 0$ and so $\beta_1(K)$ is finite.*

Proof. The first part follows because if $G/K = Q$ then $\beta_1(G) \leq \beta_1(K) + \beta_1(Q)$. For the second part, we take a presentation such that the generator x_1 appears with exponent sum zero in all the relators r_i. This can always be achieved by Tietze transformations and we say that such a presentation is in standard form with respect to x_1. Note that consequently the images of the x_j in $\overline{G}$ must have finite order for $j \geq 2$. On forming the Alexander matrix

we see that all entries $D_1(r_i)$ in the first column are zero; otherwise we put $t = f(x_1)$ equal to 1 so that $D_j(r_i)$ becomes the exponent sum matrix of the presentation. Hence if $\Delta_G(1) = 0$ then all minors evaluated by deleting the first column are zero, thus any $n-1$ relators are linearly dependent when abelianised, meaning that $\beta_1(G) \geq 2$.

$\square$

The next point is the crucial fact which allows us to use the Alexander polynomial to detect largeness.

Theorem 2.3 *If G is a finitely presented group with a homomorphism χ onto $\mathbb{Z}$ such that $\Delta_{G,\chi} = 0$ then G is large.*

Proof. This is obtained by examining Howie's condition for largeness in [27] Section 2. Adopting that notation, we let K be the standard connected 2-complex obtained from a finite presentation of G consisting of n generators and m relators, with $N = \ker \chi$ and $\overline{K}$ the 2-complex which is the regular covering of K corresponding to N so that $\pi_1(\overline{K}) = N$. Let F be a field: on following through the proof of [27] Proposition 2.1, we see that if $H_1(\overline{K}; F)$ contains a free $F[\mathbb{Z}]$-module of rank at least 1 then the conclusion of the proposition holds. But this is the hypothesis of [27] Theorem 2.2 which proves that for any sufficiently large n the finite index subgroup NG^n admits a homomorphism onto the free group of rank 2.

In our case we have that $H_1(\overline{K}; \mathbb{Z})$ is a module over $\mathbb{Z}[t^{\pm 1}]$ where t is the generator of $\mathrm{Im}(\chi)$ and acts by conjugation on $H_1(\overline{K}; \mathbb{Z})$. The process of forming the Alexander matrix A with respect to χ using the free differential calculus results in a presentation matrix P for the module $H_1(\overline{K}; \mathbb{Z}) \oplus \mathbb{Z}[t^{\pm 1}]$ (which is an $n \times m$ matrix as P is actually the transpose of A), with the second term in this direct sum meaning that we can make a basis change to P so that the bottom row is zero (see [32] page 117 for details). Thus $H_1(\overline{K}; \mathbb{Z})$ is a finitely presented $\mathbb{Z}[t^{\pm 1}]$-module. Now let us move to rational coefficients, thus taking $F = \mathbb{Q}$ in Howie's result above. We still have that $H_1(\overline{K}; \mathbb{Q})$ is a finitely presented module over $\mathbb{Q}[t^{\pm 1}]$ but this is a principal ideal domain, so by the structure theorem it is a direct sum of cyclic modules. Thus the presentation matrix P can be put into canonical form in which all off-diagonal entries are 0, as well as the bottom row. The Alexander polynomial $\Delta_{G,\chi}$ (now having rational coefficients but still of the same degree) is just the product $p_1 \ldots p_{n-1}$ where $p_i \in \mathbb{Q}[t^{\pm 1}]$ are the diagonal entries, with $p_n = 0$. This is because the elementary ideals are invariant, but the $(n-1) \times (n-1)$ minors are all zero unless the bottom row is crossed out. As $\Delta_{G,\chi} = 0$, we

must have $p_i = 0$ for some $i < n$ so $H_1(\overline{K};\mathbb{Q})$ does have a free $\mathbb{Q}[\mathbb{Z}]$-module of rank 1 in its decomposition.

$\square$

Corollary 2.4 *If G is a finitely presented group possessing a homomorphism to $\mathbb{Z}$ with kernel having infinite Betti number then G is large.*

Proof. This is just Proposition 2.1 and Theorem 2.3. In fact Theorem (B) of the Digression in [25] Section 4.5 has a similar statement with bounded cohomology.

$\square$

Note: 1. If $n - m \geq 2$ then there are no $(n-1)\times(n-1)$ minors, whence we define the Alexander ideal (and consequently the Alexander polynomial) to be zero. Thus Theorem 2.3 also includes the "deficiency at least 2 implies large" result.
2. The Corollary is most definitely not true for all finitely generated groups; we do require a finite number of relators too. Let G be the restricted wreath product $\mathbb{Z}\wr\mathbb{Z}$ which has presentation

$$\langle x_i, y | x_i = y^i x_0 y^{-i}, x_0 x_i x_0^{-1} = x_i\rangle \qquad \text{for } i \text{ in } \mathbb{Z}.$$

This is generated by $x = x_0$ and y and is soluble (with G' abelian, although it is infinitely generated) so is a long way from being large. But the homomorphism χ with $\chi(x) = 0, \chi(y) = 1$ has as its kernel the direct product of $\mathbb{Z}$ copies of $\mathbb{Z}$.

Of course the converse of Corollary 2.4 is not true in general because we certainly have finitely presented groups G which are large but with $\beta_1(G) \leq 1$, so $\Delta_{G,\chi} \neq 0$ by Proposition 2.2. However such a G will have a finite index subgroup H, for which we write $H \leq_f G$, that surjects to the non-abelian free group F_2 of rank 2 and certainly $\beta_1(H) \geq 2$. Thus we must ask whether Corollary 2.4 recognises largeness when we are "staring a free group in the face" and indeed it does.

Proposition 2.5 *If H is a finitely presented group which has a surjective homomorphism θ to a non-abelian free group F_n of rank $n \geq 2$ then we have homomorphisms χ from H onto $\mathbb{Z}$ with $\Delta_{H,\chi} = 0$.*

Proof. There are homomorphisms χ onto $\mathbb{Z}$ which factor through F_n; take any one of these so that $\chi = \tilde{\chi}\theta$. Then θ sends ker χ onto ker $\tilde{\chi}$, but the free

group F_n has no non-trivial finitely generated normal subgroups of infinite index, so ker $\tilde{\chi}$ is an infinitely generated free group with $\beta_1(\ker\tilde{\chi}) = \infty$. Thus $\beta_1(\ker\chi) = \infty$ and $\Delta_{H,\chi} = 0$ by Proposition 2.1.

□

If we regard the space of homomorphisms from G to $\mathbb{Z}$ as $\mathbb{Z}^{\beta_1(G)}$ then we can think of Proposition 2.5 as saying that we must have at least a whole 2 dimensional "subspace" of homomorphisms χ with $\Delta_{G,\chi} = 0$ for G to surject to a non-abelian free group (and this is not even sufficient; see Example 2.7) whereas Theorem 2.3 only requires a 1 dimensional subspace for G to be large. Thus the idea is that Theorem 2.3 should be able to detect largeness "at a distance", given that we do not need to find a specific finite index subgroup H of G surjecting to F_2 if we just want to establish that G is large.

Theorem 2.6 *There is an algorithm which, on being given any finite presentation as input, is guaranteed to terminate with the answer yes if and only if the group G defined by that presentation is large (but which might not terminate if G is not large).*

Proof. Recall that there is an algorithm which takes as input a finite presentation and a positive integer n and which outputs all the (finitely many) subgroups H having index n in the group G defined by the presentation. This is shown in [17] and is based on the Todd-Coxeter coset enumeration process. The output for each H is a list of generators of H and a coset table for the right regular action of G on the cosets of H. This allows us by the Reidemeister-Schreier rewriting process to give a finite presentation for H. Thus our algorithm works by taking each n in turn and each H of index n and using the Fox derivatives to form the Alexander matrix, thus enabling us to find the Alexander polynomial Δ_H (which is the highest common factor of elements in the unique factorisation domain $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$ so we can calculate it using [14] Algorithm 3.2.10). We now wish to see whether there is a homomorphism χ from H onto $\mathbb{Z}$ with $\Delta_{H,\chi} = 0$ which would prove that H, and hence G, is large by Theorem 2.3. For this we require $\beta_1(H) \geq 2$ by Proposition 2.2, so we move on if $\beta_1(H) \leq 1$. Otherwise we have infinitely many χ but Lemma 2.7 below shows that given Δ_H, it is a finite process to determine whether there is χ with $\Delta_{H,\chi} = 0$. If there is then we stop with the answer G is large. Otherwise we move on to the next finite index subgroup, and if G is large we will eventually find an H with a surjective homomorphism to F_2 so by Proposition 2.5 we will have $\Delta_{H,\chi} = 0$ and we will stop with the answer yes.

If G is finite with order N then when $n = N$ we are just performing Todd-Coxeter coset enumeration with the identity subgroup, which is guaranteed to terminate and give this order so our algorithm stops too and would output: "No, G is not large". However if G is infinite but not large then it is clear that the algorithm will run for ever (even if G has no proper subgroups of finite index, as we still need to run the finite index subgroups algorithm for each n to confirm there are no subgroups of index n).

□

One could arrange the algorithm so that in certain special cases it recognises an infinite group G which is not large, for instance it might notice that all generators commute and so G is abelian, allowing it to output the answer no. However this will not determine whether a finitely presented group is large or not for all inputs. It might be assumed that this problem is provably insolvable, as is the case for so many group theoretic properties. However this is not immediate and is an interesting question. The two standard methods used in establishing unsolvability are to show that the property is Markov or is incompatible with free products (see [35] Chapter IV Section 4). Now every non-large finitely presented group can be embedded in a large finitely presented group and vice versa (for we can embed any finitely presented group in one which has no proper finite index subgroups by [7] Part III Chapter Γ Proposition 7.7). Moreover largeness and non-largeness are both easily seen to be compatible with free products. If we call Theorem 2.6 a "half" or "partial" algorithm in that it is proved to recognise largeness but may not terminate when faced with the complementary property, we remark that unsolvable properties can have half algorithms, for instance triviality and finiteness (both Markov properties) are recognised by Todd-Coxeter coset enumeration and a half algorithm for word hyperbolicity (also a Markov property as word hyperbolic groups cannot contain $\mathbb{Z} \times \mathbb{Z}$) is given in [40]. In addition it is noted in [26] that there is a (computationally inefficient) half algorithm for a finitely presented group to have a homomorphism onto a non-abelian free group (which is not relevant to our half algorithm because it would not terminate if there is no homomorphism, and if there is then we would pick up largeness more quickly by Proposition 2.5 and Theorem 2.3) but it is not known if there is a half algorithm for the non-existence of such a homomorphism.

Example 2.7

Suppose $G = \langle x, y | r \rangle$ is a 2-generator 1-relator group such that the exponent

sum of x in r is zero (which can always be arranged by Tietze transformations). A quick way to calculate $\Delta_{G,\chi} \in \mathbb{Z}[x^{\pm 1}]$ for χ the homomorphism sending x to 1 and y to 0 is to write r in terms of $y_i = x^i y x^{-i}$ for $i \in \mathbb{Z}$ and then abelianise, with each appearance of y_i contributing a term x^i. This can be done quickly by drawing out the relation a letter at a time, where an appearance of x moves us up a level (and down one for x^{-1}), and we record the number of letters y, with their sign, that appear at each level thus giving us the coefficients of $\Delta_{G,\chi}(x)$.

Now suppose $\beta_1(G) = 2$ so that the exponent sum of y in r is zero too. We see that $\Delta_{G,\chi}(x) = 0$ if and only if the exponent sum of y at each level is zero. By taking any r for which this holds, we obtain many 2-generator 1-relator groups which are large. Assuming that r is non-trivial, these 2-generator groups cannot surject to F_2 directly. If we take $r \in F_2''$ then $\Delta_G(x,y) = 0$ and so $\Delta_{G,\chi} = 0$ for all χ.

Example 2.8

We give a brief outline of how the algorithm allows us to show that a particular 2-generator 2-relator group is large. The group $G = \langle x, y \rangle$ is the fundamental group of the closed hyperbolic 3-manifold v1539(5,1) as taken from the census [13]; it is the only example where $\beta_1(G) \geq 2$. On getting Magma to take the given presentation and calculate the abelianisation of the low index subgroups of G, we see two subgroups of index 5 with abelianisation $\mathbb{Z}^6$. On requesting a presentation of these subgroups, we find that the first has six generators x^2y, xyx, yx^2, y^2x^{-1}, $x^{-1}y^2$, $y^{-1}xy^{-1}$ and seven relators. We start to form the Alexander matrix and in doing so send various generators to the identity in order to simplify the terms. This leads us to χ with $\Delta_{G,\chi} = 0$. In fact we then see that our subgroup surjects to the free group generated by xyx and $y^{-1}xy^{-1}$ by setting all other generators to 1.

We make a few points on how the algorithm might be implemented before finishing this section with the lemma we need. Firstly it would be more efficient just to look at one subgroup from each conjugacy class and this is exactly how the low index subgroup algorithm operates in Magma and GAP. It was noted in Theorem 2.6 that we can only get a positive answer with a subgroup H where $\beta_1(H) \geq 2$. As $\beta_1(H) \geq \beta_1(G)$ for $H \leq_f G$, this is always the case if $\beta_1(G) \geq 2$. If $\beta_1(G) \leq 1$ then these packages can calculate the abelianisation $\overline{H}$ without needing to rewrite in order to obtain a full

presentation for H because they abelianise the relations as they go along. Thus it is probably best to only allow a finite presentation to be input if it has first Betti number at least 2, and to run an initial program to find such a finite index subgroup otherwise. Of course this program might not terminate but if the group is large then it will.

Returning to the question of determining from the full Alexander polynomial Δ_G whether G has homomorphisms χ with $\Delta_{G,\chi} = 0$, we have that Δ_G is an element of $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_b^{\pm 1}]$ where $b = \beta_1(G) \geq 2$. It is helpful here to adopt the approach of [36] and [19] where we think of Δ_G as a finite subset of lattice points in $\mathbb{Z}^b$, with each point weighted by a non-zero integer obtained by regarding each monomial that appears in Δ_G with a non-zero coefficient as a lattice point, and the coefficient as the weight. The ambiguity of units just means that we can shift Δ_G by unit translations.

Lemma 2.9 *Given a finitely presented group G with $\beta_1(G) \geq 2$, there is an algorithm which determines whether or nor there exists a homomorphism χ from G onto $\mathbb{Z}$ with $\Delta_{G,\chi} = 0$.*

Proof. Given any homomorphism χ from G onto $\mathbb{Z}$, we can evaluate Δ_G at χ which means that we consider the one variable polynomial $\Delta_G(t^{k_1}, \ldots, t^{k_b})$ where $\chi(t_i) = k_i$. As this is obtained by taking highest common factors in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_b^{\pm 1}]$ and then evaluating, whereas the process for $\Delta_{G,\chi}$ is the other way round, they are both zero or non-zero together. We picture evaluation of Δ_G at χ in the following way: on factoring χ through $\mathbb{Z}^b$ we have $\tilde{\chi} : \mathbb{Z}^b \to \mathbb{Z}$ which we extend to an affine map $\phi : \mathbb{R}^b \to \mathbb{R}$. Then for $x \in \mathbb{R}$ we know that $\phi^{-1}(x)$ is a hyperplane and Δ_G is zero on evaluation precisely when the following condition is satisfied: for all $m \in \mathbb{Z}$ with $\phi^{-1}(m) \cap \Delta_G \neq \emptyset$, we require that the sum of the weights corresponding to the points of Δ_G in this hyperplane $\phi^{-1}(m)$ is zero. Let us refer to this situation as "Δ_G cancels along parallel hyperplanes of constant χ".

We let C be the convex hull of Δ_G in $\mathbb{R}^b$. As Δ_G is non-empty (or else we have largeness immediately) and finite, C is a convex polytope (sometimes called the Newton polytope of the polynomial) and we refer to [9] for details. We describe our algorithm by induction on the dimension and assume it for values less than b. First if we are to have $\Delta_G(t^{k_1}, \ldots, t^{k_b}) = 0$ for some χ then this holds on putting $t = 1$ so we should check that the sum of weights over all points in Δ_G is zero and if not we are done for all χ with answer no. If so then as well as saving time, this allows us to assume that the dimension of C is b as otherwise C is contained in a hyperplane H and we have cancellation

purely within H, giving the answer yes. Now if there is some χ such that Δ_G cancels along parallel hyperplanes of constant χ then there must be a proper supporting hyperplane S for C from this family, giving rise to a (proper exposed) face $S \cap C$ of C. This cannot just be a vertex because C being the convex hull of Δ_G means the vertices of C are contained in Δ_G but the non-zero weight on one vertex would have nothing else to cancel out with in S, thus $S \cap C$ is also a convex polytope of dimension at least 1.

Thus we proceed by considering each 1 dimensional face (edge) of C in turn and asking whether it can lie in a supporting hyperplane S giving rise to cancellation. We check whether the sum of weights in our edge F_1 is zero. If not then such an S must intersect C in a 2 dimensional face F_2 containing F_1 so we replace F_1 with F_2 and repeat. Either we reach F_{b-1} which now completely determines S, so we can check directly for cancellation, or we have F_k with the weights in $\Delta_G \cap F_k$ having zero sum. This does not completely determine the possible supporting hyperplane but we deal with this by taking the quotient vector space $Q = \mathbb{R}^b/U$ of dimension $b-k$, where U is the subspace of dimension k which is a translation of the affine subspace generated by F_k. We then use the quotient map to regard Δ_G as a finite subset of Q, with new weights obtained by summing within the translates of U. Note that Δ_G cancels along hyperplanes in $\mathbb{R}^b$ parallel to a supporting hyperplane containing F_k if and only if Δ_G cancels along parallel hyperplanes in Q, because U would lie in such a hyperplane. But by taking the convex hull of Δ_G in Q we have reduced the dimension, so we are done by induction. $\square$

3 Non-hyperbolic free-by-cyclic groups are large

The deficiency of a finite presentation is the number of generators minus the number of relators and the deficiency $def(G)$ of a finitely presented group G is the maximum deficiency over all presentations. (It is bounded above by $\beta_1(G)$ so is finite.) We have already seen that groups of deficiency at least 2 are large so it seems reasonable to ask whether we can use our algorithm to obtain large groups with lower deficiencies; clearly groups of deficiency 1 would seem like the right place to start. In fact this turns out to be a very fruitful choice, both from the point of view that calculating the Alexander polynomial of a deficiency 1 group is more efficient than for lower deficiencies, and because of the behaviour of deficiency in finite covers. To explain this,

first note that a group of lower deficiency such as the modular group $\mathbb{Z}_2 * \mathbb{Z}_3$ can have a finite index subgroup with deficiency at least 2 (in this case a free group) and so be proved large in this way. Thus it makes sense to consider the virtual deficiency $vdef(G)$ of G which is the supremum of $\{def(H) : H \leq_f G\}$. It is clear that $vdef(G) \geq 2$ implies that G is large, but it is a good idea to divide up the possibilities into three distinct cases: $vdef(G) \geq 2$, $vdef(G) = 1$ and $vdef(G) \leq 0$. On taking a presentation for G with n generators and m relators where $def(G) = n - m$, we can use Reidemeister-Schreier rewriting to obtain a presentation for an index i subgroup H of G with $(n-1)i + 1$ generators and mi relators, thus the deficiency of H is at least $(def(G) - 1)i + 1$. This means that $vdef(G) \geq 2$ is equivalent to $vdef(G) = \infty$ in which case the deficiency of a finite index subgroup tends to infinity with the index. Moreover if $def(G) = 1$ then either $vdef(G) = 1$ with $def(H) = 1$ for all $H \leq_f G$ or G is large with $vdef(G) = \infty$.

The saving we gain with the algorithm in Theorem 2.6 when $def(G) = 1$ and $\beta_1(G) = b \geq 2$ is that on inputting a presentation of G with n generators and $n-1$ relators, it appears that we need to remove n columns and calculate n minors M_i, where we denote by M_i the minor with the ith column removed, then take their highest common factor in order to calculate Δ_G. In fact we can use the proof of [12] Theorem 3.1 which shows that for $1 \leq j, k \leq n$ we have

$$M_k(1 - f(x_j)) = M_j(1 - f(x_k))$$

where f is the natural ring homomorphism from $\mathbb{Z}[F_n]$ to $\mathbb{Z}[ab(G)]$ via $\mathbb{Z}[G]$. In particular, if the image of the generator x_j has finite order in the abelianisation $\overline{G}$ then there is no point in calculating M_j as it is 0, whereas if $f(x_j) \neq 0$ we can take x_k such that $1 - f(x_k)$ is coprime to $1 - f(x_j)$ in $\mathbb{Z}[ab(G)]$. Thus $1 - f(x_j)$ divides M_j, giving $M_j = (1 - f(x_j))\delta$ with δ independent of j, hence δ is the highest common factor of the minors and so is Δ_G. The same also applies if we wish to calculate $\Delta_{G,\chi}$ for some particular homomorphism χ, except that now we have $M_j = (1 - t^{n_j})\Delta_{G,\chi}(t)/(1-t)$ where $n_j = \chi(x_j)$. Moreover we will be able to obtain deficiency 1 presentations for all finite index subgroups H of G, and even if it happens that there is an H with $def(H) \geq 2$ then although the deficiency might not be picked up using the algorithm, we will find on calculating Δ_H that it is zero so we will obtain largeness.

A wide and important class of deficiency 1 groups is obtained by taking a free group F_n with free basis $x_1, \ldots, x_n$ and an automorphism α of F_n to

create the mapping torus G with presentation

$$\langle x_1,\ldots,x_n,t|tx_1t^{-1}=w_1,\ldots,tx_nt^{-1}=w_n\rangle$$

where $w_i=\alpha(x_i)\in F_n$ and thus $w_1,\ldots,w_n$ also forms a free basis. Equivalently G is a semidirect product $F_n\rtimes\mathbb{Z}$ which is the same as being (finitely generated free)-by-$\mathbb{Z}$ and implies that G is finitely generated (free-by-$\mathbb{Z}$). The following facts about such groups are well known and are summarised here.

Proposition 3.1 *Let G be as above then*
(i) Each element of G has a unique expression of the form kt^i where $k\in F_n$, and we multiply by the rule $k_1t^{i_1}k_2t^{i_2}=k_1\alpha^{i_1}(k_2)t^{i_1+i_2}$.
(ii) For each $j\in\mathbb{N}$ we have the cyclic cover $G_j=\langle F_n,s=t^j\rangle$ of index j in G with presentation

$$\langle x_1,\ldots,x_n,s|sx_1s^{-1}=\alpha^j(x_1),\ldots,sx_ns^{-1}=\alpha^j(x_n)\rangle.$$

(iii) If $H\leq_f G$ then H is also a mapping torus of an automorphism of the finitely generated free group $H\cap F_n$ which has finite index in F_n.
(iv) As well as those in (ii) we have many more finite index subgroups (assuming F_n is not the trivial group F_0). For instance given $F_m\leq_f F_n$ of index d, α acts on the (finitely many) subgroups of index d, thus on taking α^j which fixes F_m we have $H=\langle F_m,t^j\rangle$ which is of index dj. There are other finite index subgroups but they all contain a subgroup of this form. By using the fact that F_n is residually finite, we immediately obtain from (i), (ii) and (iv) the well known result that G is residually finite.

That G has deficiency no higher than 1 can be seen in a variety of ways and we briefly mention three: the 2-complex associated to such a presentation is aspherical so we can take the Euler characteristic; the Alexander polynomial Δ_G is non-zero because it must divide $\Delta_{G,\chi}$, where χ is the natural homomorphism associated with the automorphism α given by $\chi(t)=1$ and $\chi(x_i)=0$; or note that Proposition 3.1 (ii) gives us subgroups G_j of arbitrarily high index but with a bounded number of generators, so $\beta_1(G_j)$ and thus $def(G_j)$ are bounded but this would contradict the point about $vdef(G)$ if $def(G)\geq 2$. Consequently $vdef(G)=1$ by Proposition 3.1 (iii).

We now present the crucial point which allows us to gain largeness from our algorithm in many cases of groups with deficiency 1.

Theorem 3.2 *If G is a group with a deficiency 1 presentation $\langle x_1, \ldots, x_n | r_1, \ldots, r_{n-1}\rangle$ where one of the relators is of the form $x_i x_j x_i^{-1} x_j^{-1}$ then G is large if the subgroup of $ab(G)$ generated by the images of x_i and x_j has infinite index.*

Proof. Note that the form of the relevant relator forces $\beta_1(G) \geq 2$ and gives rise to a row in the Alexander matrix where there are only two non-zero entries; these are of the form $1 - f(x_j)$ in the ith column and $f(x_i) - 1$ in the jth column (we can of course assume that $i \neq j$). To calculate $\Delta_{G,\chi}(t)$ for some homomorphism χ from G onto $\mathbb{Z}$ we evaluate the terms of the Alexander matrix using χ. We then cross out a particular column, but no rows, and work out the relevant minor. But there exists χ from G onto $\mathbb{Z}$ with both x_i and x_j in the kernel, and on evaluating our row corresponding to the special relator, our entries become $1 - t^{\chi(x_j)}$ and $t^{\chi(x_i)} - 1$ which are both zero. Thus this row of zeros means that all minors are zero, so we have $\Delta_{G,\chi}(t) = 0$ and hence largeness of G.

□

Corollary 3.3 *If $G = \langle x_1, \ldots, x_n, t\rangle$ is a mapping torus of the free group F_n with respect to the automorphism α such that α fixes a generator x_j of F_n, then G is large if $\beta_1(G) \geq 3$ or if x_j has finite order in homology.*

Proof. We have a relation $t x_j t^{-1} x_j^{-1}$ in a deficiency 1 presentation for G so that $\beta_1(G) \geq 2$ and Theorem 3.2 applies unless t and x_j generate the homology up to finite index.

□

Of course a group G with a presentation having a relator of the form $xyx^{-1}y^{-1}$ is not equivalent to G containing a $\mathbb{Z} \times \mathbb{Z}$ subgroup even in the deficiency 1 case: if such a relation is present, other relations could imply that x say has finite order (as in $(\mathbb{Z}_2 \times \mathbb{Z}) * \mathbb{Z}$). As for the other way round, if $\beta_1(G) = 1$ then it cannot have such a relation, so a $\mathbb{Z} \times \mathbb{Z}$ subgroup cannot always be "promoted" to a defining relation without worsening the deficiency of the presentation. What we can do in the case of a free group automorphism mapping torus is promote any $\mathbb{Z} \times \mathbb{Z}$ subgroup to a relation in a finite index subgroup H and this is how we will proceed. We are then done provided H has enough homology to apply Corollary 3.3 and we can obtain this by again moving to finite index subgroups, first so as to gain finite order homology and then we can use this to increase the first Betti number.

Hence let G be a mapping torus of an automorphism α of the free group F_n. We say that α has no periodic conjugacy classes if whenever $\alpha^j(w)$ is conjugate in F_n to w for $w \in F_n$, we have $w = 1$ or $j = 0$. Taken together, the results of [4], [5] and [8] show that this is equivalent to G containing no subgroups isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and also to G being word-hyperbolic. Thus we are proving here the largeness of the non-word-hyperbolic mapping tori of automorphisms of F_n and we leave open the question of whether we have largeness in the word-hyperbolic case, which is perhaps surprising because it might be thought that the well developed theory of word-hyperbolic groups would mean they are the more tractable case. However we point out that in the proof we do not refer to word-hyperbolicity at any point; one uses existence of the periodic conjugacy class directly. We show here that the first two conditions are equal because it can be established quickly by elementary means.

Lemma 3.4 *The mapping torus G of an automorphism α of the free group F_n contains $\mathbb{Z} \times \mathbb{Z}$ if and only if α has a periodic conjugacy class.*

Proof. If there is $w \neq 1$ and $j \neq 0$ with $\alpha^j(w) = vwv^{-1}$ then the automorphism $\gamma_v^{-1}\alpha^j$ fixes w, where we use γ_v to denote the inner automorphism of F_n that is conjugation by v. Thus the mapping torus of F_n with respect to $\gamma_v^{-1}\alpha^j$ contains $\mathbb{Z} \times \mathbb{Z}$ because if conjugation by s_0 denotes the image of $\gamma_v^{-1}\alpha^j$ then w and s_0 commute, with $w^{i_1}s_0^{i_2} = 1$ implying that $i_1 = i_2 = 0$ by Proposition 3.1 (i). But changing α^j to $\gamma_v^{-1}\alpha^j$ does not alter the cyclic cover G_j as a mapping torus because it is just replacing $s = t^j$ with $s_0 = v^{-1}s$ in the presentation given by Proposition 3.1 (ii). Hence $\mathbb{Z} \times \mathbb{Z} \leq G_j \leq G$.

Now suppose $G = \langle F_n, t\rangle$ has two elements $x = kt^i$, $y = lt^j$ for $k, l \in F_n$ which generate $\mathbb{Z} \times \mathbb{Z}$. We can assume $i \neq 0$ as we cannot have both x and y in F_n, and on taking $z = x^jy^{-i} \in F_n \setminus \{1\}$ we have that $xzx^{-1} = z$ implies $\alpha^i(z) = k^{-1}zk$.

□

Thus our mapping torus G which contains $\mathbb{Z} \times \mathbb{Z}$ can be assumed to have a non-trivial element $w \in F_n$ with $\alpha(w) = w$ by dropping down to a finite cover and using conjugation to change the automorphism.

Proposition 3.5 *If α is an automorphism of F_n having $w \in F_n \setminus \{1\}$ with $\alpha(w) = w$ and $G = \langle F_n, t\rangle$ is the associated mapping torus then there is a finite index subgroup $H = \langle F_m, t^j\rangle$ of G where $F_m \leq_f F_n$ has a free basis which includes w.*

Proof. We use the classic result of Marshall Hall Jnr. that if L is a non-trivial finitely generated subgroup of the non-abelian free group F_n then there is a finite index subgroup F_m of F_n such that L is a free factor of F_m. We just need to put $L = \langle w \rangle$ so that $F_m = \langle w \rangle * C$ for some $C \leq F_n$ with w a basis element for F_m. Now we take $j > 0$ with α^j fixing F_m as in Proposition 3.1 (iv) and this gives us $H = \langle F_m, s = t^j \rangle$ whose natural presentation has deficiency 1 and contains the relation $sws^{-1} = w$.

□

We are now in the position to apply Corollary 3.3 to H if $\beta_1(H) \geq 3$, obtaining largeness. But if $\beta_1(H) \geq 2$ then we need to take further finite index subgroups in order to gain more homology. The "smallest" possible abelianisation of H is $\mathbb{Z} \times \mathbb{Z}$ in which case we need to proceed in two steps. It is only now where we require that our free group F_n is non-abelian so that $n \geq 2$. If $n = 1$ then we will have reached this point with H itself equal to $\mathbb{Z} \times \mathbb{Z}$, but clearly all subgroups are of this form and H is not large.

Lemma 3.6 *If G is the mapping torus of an automorphism α of the non-abelian free group F_n with abelianisation $\overline{G} = \mathbb{Z} \times \mathbb{Z}$ then there is a cyclic cover G_j of G with its abelianisation $\overline{G_j}$ having a surjective homomorphism to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_m$ for some $m \geq 2$.*

Proof. The automorphism α induces an automorphism $\overline{\alpha}$ of $\mathbb{Z}^n$ which is an element of $GL(n, \mathbb{Z})$. Take any prime p and regard $\overline{\alpha}$ as an element of the finite group $GL(n, \mathbb{Z}_p)$ then, on taking j with $\overline{\alpha}^j = 1$, the cyclic cover G_j has a homomorphism onto the mapping torus of $\mathbb{Z}_p \times \ldots \times \mathbb{Z}_p$ formed from the identity automorphism. Thus G_j maps onto $\mathbb{Z} \times \mathbb{Z}_p \times \mathbb{Z}_p$, as well as onto $\mathbb{Z} \times \mathbb{Z}$ because $\beta_1(G_j) \geq \beta_1(G)$.

□

We can now get a finite index subgroup with first Betti number at least 3 by using Reidemeister-Schreier rewriting with respect to abelian covers.

Proposition 3.7 *If $G = \langle x_1, \ldots, x_n, t | r_1, \ldots, r_n \rangle$ has a deficiency 1 presentation with a relator $r_1 = tx_1t^{-1}x_1^{-1}$ and the abelianisation $\overline{G} = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_m$ for $m \geq 2$ then G is large.*

Proof. We are done by Theorem 3.2 unless the images $\overline{x_1}, \overline{t}$ in $\overline{G}$ generate a finite index subgroup S of $\overline{G}$, but if so then S can only be isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Take a homomorphism θ from G onto $\mathbb{Z}_j$ for $j \geq 2$ such that S is in the kernel. We require another generator $g \in \{x_2, \ldots, x_n\}$ such that $\theta(g)$

generates Im θ but this can be achieved by Tietze transformations amongst $x_2, \ldots, x_n$. We now perform Reidemeister-Schreier rewriting to obtain from our original presentation of G a deficiency 1 presentation for ker θ consisting of $nj+1$ generators and nj relators. We have g^i, $0 \leq i < j$ as a Schreier transversal for ker θ in G and on setting $t_i = g^i t g^{-i}$ and $x_{1,i} = g^i x_1 g^{-i}$, which will all be amongst the generators for our presentation of ker θ given by this process (because $t, x_1 \in S \leq$ ker θ), our original relator r_1 gives rise to j relators $t_i x_{1,i} t_i^{-1} x_{1,i}^{-1}$ in the presentation for our subgroup. As these disappear when we abelianise, we see that β_1(ker θ) is at least $j+1$ and we are done by Theorem 3.2.

□

We have now covered all cases of finitely generated non-word-hyperbolic free-by-cyclic groups.

Theorem 3.8 *If G is a finitely generated group which is F-by-$\mathbb{Z}$ for F free then G is large if F is infinitely generated or if G contains $\mathbb{Z} \times \mathbb{Z}$, with the sole exception of $F = \mathbb{Z}$ and $G = \langle x, y | xyx^{-1} = y^{\pm 1} \rangle$.*

Proof. We have by [23] that if G is finitely generated then it is finitely presented, even if F is infinitely generated. If this is so then G is large by Corollary 2.4. If F is finitely generated then G is a mapping torus of an automorphism α of F_n which contains $\mathbb{Z} \times \mathbb{Z}$ and on applying Lemma 3.4 and Proposition 3.5 whilst dropping to the appropriate finite index subgroups, we can assume that G has a relator of the form $t x_1 t^{-1} x_1^{-1}$ in a deficiency 1 presentation and thus $\beta_1(G) \geq 2$. If $\beta_1 \geq 3$ then apply Corollary 3.3 directly, whereas if $\beta_1(G) = 2$ then use Lemma 3.6 if necessary, noting that the cyclic cover will preserve the form of the relator, and then Proposition 3.7.

□

One problem that will be encountered if trying to extend this result to word-hyperbolic groups of the form F_n-by-$\mathbb{Z}$ is that it would imply they have finite index subgroups with first Betti number at least two. This is unknown as we have a question of Casson which appears in various problem lists: "Does every automorphism $\alpha : F_n \to F_n$ leave invariant a finite index subgroup K such that $\alpha_{ab} : K/K' \to K/K'$ has an eigenvalue which is a root of unity" is asked in [3] Question 12.16 whereas in [38] (F33) all eigenvalues being roots of unity are required. Note that the weaker version is equivalent to our question because if G is F_n-by-$\mathbb{Z}$ with $L \leq_f G$ having $\beta_1(L) \geq 2$ then we can drop down to a subgroup $H \leq_f L$ of the form in Proposition 3.1 (iv), with $H = \langle F_m, t^j \rangle$ where $F_m \leq_f F_n$. Now take $K \leq_f F_m$ which is characteristic

in F_n, then α leaves K invariant with $\langle K, t^j\rangle \leq_f L$ having first Betti number at least 2, hence α^j_{ab} restricted to K/K' has 1 as an eigenvalue. At least we see Casson's question is true in the non-word-hyperbolic case.

However we can say something for all free-by-cyclic groups if we relax our notion of large somewhat. Recall that a finitely generated group G is said to satisfy the Tits alternative if it contains a non-abelian free group or is virtually soluble, although some stern individuals insist that for a proper Tits alternative every finitely generated subgroup of G must fall into this dichotomy.

Corollary 3.9 *If G is a finitely generated group that is free-by-cyclic then either G is SQ-universal or G is virtually abelian. The same is true of any finitely generated subgroup H.*

Proof. If G is F-by-cyclic for F free then for any $H \leq G$ we have that H is $(F \cap H)$-by-cyclic. If the cyclic quotient of G is finite then G and H are virtually free. If it is $\mathbb{Z}$ then G is large by Theorem 3.8 (thus implying SQ-universality), or G is virtually $\mathbb{Z} \times \mathbb{Z}$ or G is word-hyperbolic. But by [39] all word-hyperbolic groups are SQ-universal or virtually cyclic. As for H, either it is contained in F or it too is free-by-$\mathbb{Z}$.

□

4 Ascending HNN extensions of free groups

Having shown that mapping tori of finitely generated non-abelian free group automorphisms are large if they contain $\mathbb{Z} \times \mathbb{Z}$, we now turn to arbitrary endomorphisms. Given a homomorphism $\theta : F_n \to F_n$ we can again form the mapping torus

$$G = \langle x_1, \ldots, x_n, t | tx_1t^{-1} = \theta(x_1), \ldots, tx_nt^{-1} = \theta(x_n)\rangle$$

but we are not now assuming that θ is injective or surjective. However there is a neat way of sidestepping the non-injective case using [29] where it is noted that G is isomorphic to a mapping torus of an injective free group homomorphism $\tilde{\theta} : F_m \to F_m$ where $m \leq n$. Of course it might be that F_n is non-abelian but $m = 0$ or 1 in which case $G = \mathbb{Z}$ or $\langle a, t | tat^{-1} = a^k\rangle$ for $k \neq 0$. However in these cases G is soluble and so is definitely not large. Therefore we will assume throughout that θ is injective, in which case G

is also called an ascending HNN extension of the free group F_n, where we conjugate the base F_n to an isomorphic subgroup of itself using the stable letter t.

We also assume in this section that θ is not surjective, whereupon we call the ascending HNN extension proper, as automorphisms have already been covered in the last section. We now develop the basic facts of ascending HNN extensions of finitely generated free groups which we will find mostly mirror the case of automorphisms, but which require more care with the proofs. We have that our base $F_n = \langle x_1, \ldots, x_n \rangle$ embeds in G and we will refer to this copy of F_n in G as Γ, with $\theta(\Gamma) < \Gamma$ being isomorphic to F_n which means it has infinite index in Γ.

Once an ascending HNN extension G is formed, there is an obvious homomorphism χ from G onto $\mathbb{Z}$ associated with it which is given by $\chi(t) = 1$ and $\chi(\Gamma) = 0$. In the case of an automorphism ker χ is simply Γ but otherwise we have to consider "backwards conjugates", with

$$\ker\chi = \bigcup_{i=0}^{\infty} t^{-i}\Gamma t^{i}.$$

Note $\theta(\Gamma) = t\Gamma t^{-1} < \Gamma$ so that ker χ is a strictly ascending union of free groups, thus is infinitely generated and locally free, but never free because $\beta_1(\ker\chi) \leq \beta_1(\Gamma)$. One consequence which makes this situation rather different from the case of automorphisms is that if we are given G which we are told is a mapping torus with respect to an automorphism α of some group Γ and we are given the associated homomorphism χ then we can easily recover Γ as it is ker χ. But here we can replace Γ with $t^k\Gamma t^{-k}$ for any $k \in \mathbb{Z}$ and still have a decomposition of G as a proper ascending HNN extension with the same associated homomorphism; indeed this change does not alter the presentation which can be seen by adding new generators $y_i = t^k x_i t^{-k}$ in the presentation and then using them to replace each x_i. In fact it can be seen (say by Corollary 4.2) that any finitely generated subgroup Δ of Γ which contains $\theta(\Gamma)$ can replace Γ whilst keeping G as a proper ascending HNN extension with the same associated homomorphism (although now the presentation for G will of course change) so the base is not even defined up to isomorphism. Whilst this means that we must be careful, it also allows us to replace Γ by a more convenient base in the proofs if required.

The following result, which is Lemma 3.1 in [24], allows us to recognise ascending HNN extensions "internally".

Lemma 4.1 *A group G with subgroup Γ is an ascending HNN extension with Γ as base if and only if there exists $t \in G$ with*
(1) $G = \langle \Gamma, t \rangle$;
(2) $t^k \notin \Gamma$ for any $k \neq 0$;
(3) $t\Gamma t^{-1} \leq \Gamma$.

Given such an ascending HNN extension $G = \langle \Gamma, t \rangle$ where the base Γ is finitely generated, the next lemma reveals all the possible finitely generated bases for expressing G as an ascending HNN extension without changing the associated homomorphism.

Lemma 4.2 *The finitely generated subgroup Δ of $G = \langle \Gamma, t \rangle$ is a base for the decomposition of G as an ascending HNN extension with the same associated homomorphism χ if and only if $t\Delta t^{-1}$ is contained in Δ and there are integers $k \geq l$ with*

$$t^k \Gamma t^{-k} \leq \Delta \leq t^l \Gamma t^{-l}.$$

Proof. If so then conditions (1)–(3) in Lemma 4.1 apply to Δ, whereas if Δ is a base then $t\Delta t^{-1} \leq \Delta$ is required, and the associated homomorphism χ would be such that

$$\ker \chi = \bigcup_{i=0}^{\infty} t^{-i} \Delta t^i$$

but as $\Gamma \leq \ker \chi$ is finitely generated we must have k with $\Gamma \leq t^{-k}\Delta t^k$. Now swap Γ and Δ.

□

We can now offer the equivalent of Proposition 3.1 for injective endomorphisms.

Proposition 4.3 *Let G be a proper ascending HNN extension*

$$\langle x_1, \ldots, x_n, t | t x_1 t^{-1} = \theta(x_1), \ldots, t x_n t^{-1} = \theta(x_n) \rangle$$

with respect to the injective endomorphism θ of the finitely generated free group $\Gamma = F_n$ with free basis $x_1, \ldots, x_n$ and let χ be the associated homomorphism.
(i) Each element g of G has an expression of the form $g = t^{-p}\gamma t^q$ for $p, q \geq 0$.
(ii) For each $j \in \mathbb{N}$ we have the cyclic cover $G_j = \langle \Gamma, s = t^j \rangle$ of index j in G with presentation

$$\langle x_1, \ldots, x_n, s | s x_1 s^{-1} = \theta^j(x_1), \ldots, s x_n s^{-1} = \theta^j(x_n) \rangle.$$

(iii) If $H \leq_f G$ then H is also a proper ascending HNN extension of a finitely generated free group $\Delta \leq_f \Gamma$ with respect to the (restriction to H of the) same associated homomorphism χ.
(iv) If $\Delta \leq_f \Gamma$ then $H = \langle \Delta, t \rangle$ has finite index in $G = \langle \Gamma, t \rangle$.

Proof. (i) is [23] Lemma 2.2 (1). The point is that positive powers can be moved to the right and negative powers to the left, thus allowing us to multiply two elements together. However the expression is not unique as $g = t^{-p}\gamma t^q = t^{-p-1}\theta(\gamma)t^{q+1}$. Note that $\chi(g) = q - p$.
(ii) This is also Lemma 2.2 (1), but in [28].
(iii) We try following the standard proof for automorphisms, but with more care. Setting $K = \ker \chi$ we have $H/(H \cap K) = \mathbb{Z}$ so let $x \in H$ be a generator of $\mathbb{Z}$, which we can take to be of the form $t^{-p}\gamma t^{p+m}$ for $\gamma \in \Gamma$, $p \geq 0$ and $m > 0$ (where m is the minimum positive value of χ when restricted to H). We now put $t^{-p}\Gamma t^p$ in place of Γ so we can write x as γt^m. Then setting $\Delta = H \cap \Gamma$ we have that $\chi(x) = m$ and $\chi(\Delta) = 0$ so we are done if the three conditions of Lemma 4.1 are satisfied for Δ and x. We certainly have that $x^k \notin \Delta$ for $k \neq 0$. Also $x\Delta x^{-1} \leq \Delta$ because $x \in H$ and $x\Gamma x^{-1} \leq \Gamma$.

In order to get that x and Δ generate H, we show by induction that $x^{-i}\Gamma x^i$, which is contained in $t^{-im}\Gamma t^{im}$, also contains it. Assuming this to be true for i, we have

$$x^{-1}t^{-im}\Gamma t^{im}x = t^{-(i+1)m}(\theta^{im}(\gamma^{-1})\Gamma\theta^{im}(\gamma))t^{(i+1)m} = t^{-(i+1)m}\Gamma t^{(i+1)m}.$$

Now we can write K as a smaller ascending union

$$\bigcup_{i=0}^{\infty} t^{-im}\Gamma t^{im} = \bigcup_{i=0}^{\infty} x^{-i}\Gamma x^i$$

so that $H \cap K$ is the union of the $H \cap x^{-i}\Gamma x^i$ which equals $x^{-i}(H \cap \Gamma)x^i$ because $x \in H$. Thus H is generated by x and Δ. Finally we change Γ back to $t^p\Gamma t^{-p}$ so that Δ is changed to $t^p\Delta t^{-p}$ which is in the original Γ and which is also a perfectly good base for H by Lemma 4.2.
(iv) Let $\gamma_1, \ldots, \gamma_d$ be a transversal for Δ in Γ. If we can show that $H \cap K \leq_f K$ for K the kernel of the associated homomorphism then we are done, despite the fact that K and $H \cap K$ are infinitely generated, because $t \in H$ and any $g \in G$ is of the form kt^m for $k \in K$.

The set

$$S = \{t^{-m}\gamma_i t^m : m \in \mathbb{N}, 1 \leq i \leq d\}$$

contains an element of every coset of $H \cap K$ in K. This can be seen by writing $k \in K$ as $t^{-m}\gamma t^m$ for $\gamma \in \Gamma$ using (i). Then there is γ_i such that $\gamma\gamma_i = \delta \in \Delta$. This means that $kt^{-m}\gamma_i t^m = t^{-m}\delta t^m$ which is in H and in K. We now show that the index of $H \cap K$ in K is at most d. Note that for $q > p$, any element of the form $t^{-p}\gamma_i t^p$ is in the same coset as some element of the form $t^{-q}\gamma_j t^q$ because $\theta^{q-p}(\gamma_i)\gamma_j^{-1} = \delta$ for some $j \in \{1, \ldots, d\}$ and some $\delta \in \Delta$, thus giving $t^{-p}\gamma_i t^p(t^{-q}\gamma_j t^q)^{-1} = t^{-q}\delta t^q$ which is in $H \cap K$. Therefore we proceed as follows: S is a set indexed by $(l, i) \in \mathbb{N} \times \mathbb{Z}_d$ and we refer to l as the level. Choose a transversal T for $H \cap K$ in K from S which a priori could be infinite and let g_1 be the element in T with smallest level l_1 (and smallest i if necessary). Then for each level $l_1 + 1, l_1 + 2, \ldots$ above l_1 there is an element in S with this level that is in the same coset of $H \cap K$ as g_1 and so cannot be in T. Cross these elements off from S and now take the next element g_2 in T according to our ordering of S. Certainly g_2 with level l_2 has not been crossed off and we repeat the process of removing one element in each level above l_2; as these are in the same coset as g_2 they too have not been erased already. Now note that we can go no further than g_d because then we will have crossed off all elements from all levels above l_d; thus we must have a transversal for $H \cap K$ in K of no more than d elements.

□

We now consider periodic conjugacy classes in the injective endomorphism case. We follow [28] in generalising the previous definition of a periodic conjugacy class which was applied to automorphisms. Let $G = \langle F_n, t \rangle$ be the mapping torus of an injective endomorphism θ of the free group F_n. We say that θ has a periodic conjugacy class if there exists $i > 0$, $k \in \mathbb{Z}$ and $w \in F_n \backslash \{1\}$ such that $\theta^i(w)$ is conjugate to w^k in F_n. If this is so with $\theta^i(w) = vw^k v^{-1}$ then on taking the endomorphism ϕ of F_n such that $\phi = \gamma_v^{-1}\theta^i$ we have on setting $\Delta = \langle w \rangle$ and $s = v^{-1}t^i$ that the subgroup $\langle \Delta, s \rangle$ of G is an ascending HNN extension with base Δ and stable letter s by Lemma 4.1. Consequently it has the presentation $\langle s, w | sws^{-1} = w^k \rangle$. These presentations are part of the famous family of 2-generator 1-relator subgroups known as the Baumslag-Solitar groups. We define the Baumslag-Solitar group $B(j, k) = \langle x, y | xy^j x^{-1} = y^k \rangle$ for $j, k \neq 0$ and without loss of generality $|k| \geq j > 0$ because we can replace x with x^{-1} and put it on the other side, as well as taking the inverse of the relation. The Baumslag-Solitar groups are used as counterexamples in many areas of group theory so it is useful to be aware of when they can be subgroups of mapping tori of free groups. The next Proposition is similar to [28] Lemma 2.3 but has a

completely general conclusion.

Proposition 4.4 *Let $G = \langle F_n, t\rangle$ be the mapping torus of an injective endomorphism θ of the free group F_n.*
(i) G cannot contain a subgroup isomorphic to a Baumslag-Solitar group $B(j,k)$ unless $j = 1$ or $j = k$.
(ii) If there exists $i, j > 0$, $k \in \mathbb{Z}$ and $w \in F_n \backslash \{1\}$ with $\theta^i(w^j)$ conjugate in F_n to w^k then $k = dj$ and $\theta^i(w)$ is conjugate to w^d so that θ has a periodic conjugacy class.
(iii) G has Baumslag-Solitar subgroups if and only if θ has periodic conjugacy classes. In particular G contains $B(1,d)$ if and only if G has a periodic conjugacy class of the form $w \in F_n \backslash \{1\}$ and $i > 0$ with $\theta^i(w)$ conjugate to w^d.
(iv) If θ is an automorphism then G can only contain Baumslag-Solitar subgroups of the form $B(1, \pm 1)$ or $B(j,j)$. This happens if and only if G has a periodic conjugacy class of the form $w \in F_n \backslash \{1\}$ and $i > 0$ with $\theta^i(w)$ conjugate to $w^{\pm 1}$, so that our current definition of periodic conjugacy classes is equivalent to the definition given for automorphisms in Section 3.

Proof. We prove (ii) first. Suppose we have $\theta^i(w^j) = vw^kv^{-1}$ for $v \in F_n$ then we can do as we did in Lemma 3.4 and replace θ with the injective endomorphism $\gamma_v^{-1}\theta^i$, thus replacing G with a finite index subgroup of itself by Proposition 4.3 (ii). So we now have $\theta(w) = u \in F_n \backslash \{1\}$ and $\theta(w^j) = w^k$. We set $w = c^m$ where c generates a maximal cyclic subgroup. Then $(\theta(c^j))^m = (c^k)^m$ but we are in a free group, so $\theta(c^j) = c^k$. Hence $\theta(c) = c^d$ must also be a power of c as its jth power is. Thus $k = dj$ and $\theta(w) = w^d$.

For (i), if $B(j,k) \leq G$ then we have elements $x = t^{-p}at^q$, $y = t^{-r}bt^s$, with $p, q, r, s \geq 0$ and $a, b \in F_n$, which satisfy $xy^jx^{-1} = y^k$. We can assume that $q - p \geq 0$ by replacing x with x^{-1} and swapping j and k (inverting the relation if now $j < 0$). Then applying to both sides of the relation the associated homomorphism χ of the ascending HNN extension, we have $\chi(y) = 0$ or $j = k$. With the former we have $y = t^{-r}bt^r$ so by replacing $H = \langle x, y\rangle$ with the conjugate subgroup $t^N H t^{-N}$ where $N = p + r$, we obtain $y = t^p bt^{-p}$ and $x = t^r at^{q-p-r} = a't^{q-p}$ for $a' \in F_n$. Now $q - p \neq 0$ or else $H \leq F_n$ would be cyclic as it has a relation between its two generators. Thus $q - p > 0$ and $\theta^{q-p}(y^j)$ is conjugate via a' to y^k, so by (ii) $\theta^{q-p}(y)$ is conjugate via a' to y^d which implies from before that H is isomorphic to $B(1,d)$.

As for $B(j,j) = \langle x, y | xy^jx^{-1} = y^j\rangle$, note that this is a mapping torus $G = \langle F_j, y\rangle$ of a free group automorphism α of $F_j = \langle x = x_1, \ldots, x_j\rangle$ where

$\alpha(x_i) = x_{i+1} \bmod j$. However $B(1,1) = \mathbb{Z}\times\mathbb{Z} \le B(j,j)$, generated by x and y^j (as they commute and generate $\mathbb{Z}\times\mathbb{Z}$ in homology).

For (iii) we have already seen that θ having a periodic conjugacy class of the form $\theta^i(w)$ conjugate to w^d gives rise to a subgroup $B(1,d)$ of G. If $B(j,k) \le G$ then we have just said that if $j = k \ge 2$ we can replace it with $B(1,1)$ by swapping y for y^j. Thus following through the proof of (i), we have for our x and y that generate $B(j,k)$ that either $\chi(y) = 0$ whereupon we saw that $B(j,k)$ is actually $B(1,d)$ with $\theta^{q-p}(y)$ conjugate to y^d, or $j = k = 1$. In this case we set $l = q-p$ and $m = s-r$, both of which can be taken as non-negative by replacing x or y by its inverse if necessary. On moving positive and negative powers of t to the right and left respectively we have that $z = x^m y^{-l}$ is of the form $t^{-pm-sl}ct^{qm+rl}$ which as $c \in F_n$ means that $\chi(z) = 0$. So as above we have some large N such that we can replace $B(1,1)$ by $t^N B(1,1)t^{-N}$, thus we are able to regard z as if it is in F_n, and we now have $x = a't^l$ and $y = b't^m$ for $a', b' \in F_n$. Now we can take $l > 0$ because $xz = zx$ so $l = 0$ implies that x and z are in the same cyclic subgroup of F_n. But if so then swap x with y, which together generate $\mathbb{Z}\times\mathbb{Z}$. Hence θ has a periodic conjugacy class with $\theta^l(z)$ conjugate to z.

For (iv) we use the classic result of Higman which says that an automorphism of a free group that maps a finitely generated subgroup into itself maps it onto itself. Thus if θ has a periodic conjugacy class with the automorphism $\gamma_v^{-1}\theta^i$ sending w to w^d then we must have $d = \pm 1$. This happens if and only if G contains $B(1,\pm 1)$ by (iii), and we know G containing $B(j,j)$ implies that G contains $B(1,1)$. Now if $\theta^i(w) = vw^{-1}v^{-1}$ then $\theta^{2i}(w) = uwu^{-1}$ where $u = \theta^i(v)v$. Thus by (iii) G contains $\mathbb{Z}\times\mathbb{Z}$ and this is equivalent to the original definition of possessing a periodic conjugacy class.

$\square$

Hence just as in Section 3 we can assume that our mapping torus $G = \langle F_n, t\rangle$ which contains $\mathbb{Z}\times\mathbb{Z}$ has $w \in F_n \backslash \{1\}$ with $\theta(w) = w$. We would like to show that G is large by similar means where we first lift w to a generator in a free group and then create extra finite homology in order to apply Proposition 3.7. We shall see that the first task can be achieved by careful but essentially elementary means as before, whereas the second will require extra knowledge.

Theorem 4.5 *If θ is an injective endomorphism of the free group Γ of rank n with $w \in F_n \backslash \{1\}$ such that $\theta(w) = w$ then there is a finite index subgroup Δ of Γ and $j \ge 1$ such that Δ has a free basis including w, with $\theta^j(\Delta) \le \Delta$.*

Proof. The free basis is easy to obtain by taking $F \leq_f \Gamma$ with $\langle w\rangle$ a free factor of F, just as in Proposition 3.5. The second condition is the important part. The aim is to repeatedly pull back F; although we do not have $F \leq \theta^{-1}(F)$ in general as this is equivalent to $\theta(F) \leq F$ which would mean we are done, we do find that the index is non-increasing. To see this note that $\theta^{-1}(F) = \theta^{-1}(F \cap \theta(\Gamma))$ and $\theta^{-1}\theta(\Gamma) = \Gamma$ as $\theta : \Gamma \to \theta(\Gamma)$ is an isomorphism. Hence the index of $F \cap \theta(\Gamma)$ in $\theta(\Gamma)$ is preserved by applying θ^{-1} to both sides, so is equal to the index of $\theta^{-1}(F)$ in Γ. But the index of $F \cap \theta(\Gamma)$ in $\theta(\Gamma)$ is no more than that of F in Γ, thus $[\Gamma : \theta^{-i}(F)]$ gives us a non-increasing sequence which must stabilise at N with value k. When it does we have for $i \geq 0$ that $\theta^{-(i+N)}(F)$ is just moving around the finitely many index k subgroups. Although it happens that θ^{-1} is not in general a permutation of these index k subgroups, we must land on some such subgroup Δ twice so we have $j \geq 1$ with $\theta^{-j}(\Delta) = \Delta$, giving $\Delta \geq \theta^j(\Delta)$.

We now show that, although the rank of $\theta^{-i}(F)$ reduces whenever the index reduces, we can keep w as an element of a free basis each time we pull back. This time we restrict θ to an injective homomorphism from $\theta^{-1}(F)$ to F with image $\theta\theta^{-1}(F)$. As $\theta\theta^{-1}(F)$ is a finitely generated subgroup of F containing a free basis element w of F, we can ensure w is in a free basis for $\theta\theta^{-1}(F)$ (for instance see [35] Proposition I.3.19). Now $\theta^{-1}(F)$ and $\theta\theta^{-1}(F)$ are isomorphic via θ with inverse ϕ say, so a basis $b_1, \ldots, b_r$ for the latter gives rise to a basis $\phi(b_1), \ldots, \phi(b_r)$ for $\theta^{-1}(F)$ and if $b_1 = w$ then $\phi(b_1) = w$.

□

Corollary 4.6 *If $G = \langle \Gamma, t\rangle$ is a mapping torus of an injective endomorphism θ of the free group Γ of rank n and $\mathbb{Z} \times \mathbb{Z} \leq G$ then we have $H \leq_f G$ such that H has a deficiency 1 presentation $\langle x_1, \ldots, x_m, s | r_1, \ldots, r_m\rangle$ including a relator of the form $sx_1s^{-1}x_1^{-1}$.*

Proof. By Proposition 4.4 (iii) we can on dropping to a finite index subgroup of G assume that there is $w \in \Gamma\backslash\{1\}$ with $\theta(w) = w$ and then by Theorem 4.5 we have $\Delta \leq_f \Gamma$ with Δ having a free basis $w, x_2, \ldots, x_m$ and $j \geq 1$ with $\theta^j(\Delta) \leq \Delta$. Thus by Proposition 4.3 (ii) and (iv) we have that $L = \langle \Delta, s = t^j\rangle$ has finite index in G and by Lemma 4.1 L is an ascending HNN extension with base Δ and stable letter s. Thus on taking the standard presentation for L we see that it has deficiency 1 with a relator equal to $sws^{-1}w^{-1}$.

□

Thus such a G has a finite index subgroup H with $\beta_1(H) \geq 2$ and Proposition 3.7 tells us that G is large apart from in one circumstance; namely

when $\overline{H}=\mathbb{Z}\times\mathbb{Z}$. However, unlike the case of automorphisms where we used cyclic covers in Lemma 3.6 to get extra finite homology, we cannot do this for general endomorphisms with cyclic, abelian or even soluble covers as the next example shows.

Example 4.7

Let $G=\langle F_2,t\rangle$ be the mapping torus of the free group $F_2=\langle x,y\rangle$ (using capital letters for inverses) with respect to the endomorphism

$$\begin{aligned}\theta(x) &= x\\ \theta(y) &= xyXYx^2Y^2X^2y^3xYXY^2x^2y^2X^2.\end{aligned}$$

The image $w(x,y)$ of y is the commutator of the commutators $xyXY$ and $x^2Y^2X^2y^2$. As w is in $F_2'\backslash\{1\}$, it is clear that θ is injective but not surjective.

Proposition 4.8 *This group G has the property that if $N\leq_f H\leq_f G$ with $N\trianglelefteq G$ and G/N is soluble then $G'\leq N$, so that $H\trianglelefteq G$ with G/H abelian. Moreover $H/H'=\mathbb{Z}\times\mathbb{Z}$.*

Proof. We have that $\overline{G}=\mathbb{Z}\times\mathbb{Z}$ (generated by t and x) and we calculate the Alexander polynomial $\Delta_G(t,x)$. This turns out to be 1 because $w(x,y)$ has been chosen so that the group $\langle x,y|w\rangle$ has Alexander polynomial 0 as w is a commutator of commutators. By the same process we can see that the cyclic covers $G_j=\langle F_2,s=t^j\rangle$ also have $\overline{G_j}=\mathbb{Z}\times\mathbb{Z}$ and $\Delta_{G_j}=1$ because whenever we apply θ we replace in $\theta^j(y)$ each appearance of y with w, so $\langle x,y|\theta^j(y)\rangle$ also has zero Alexander polynomial. We now show that for any "rectangular" abelian cover $G_{j,k}$, namely the subgroup of index jk in G consisting of those elements with exponent sum equal to 0 modulo j in t and 0 modulo k in x, we have $\overline{G_{j,k}}=\mathbb{Z}\times\mathbb{Z}$. Using Reidemeister-Schreier to get from our presentation for G_j to that of $G_{j,k}$, we have generators $z=x^k$, s_i and y_i for $0\leq i\leq k-1$, where $s_i=x^isx^{-i}$ and equivalently for y_i. The k new relators obtained from $sxSX$ in the presentation for G_j collapse to $s_i=s$ and $szSZ$. As for the other relator r in G_j, we write it as $r=w_j(x,y)sYS$ where $w_j=\theta^j(y)$ has 0 exponent sum in y for each level of x. Then in $G_{j,k}$ we find that r becomes, when abelianised, simply Y_0 and the other relators for $G_{j,k}$ obtained from conjugates x^irX^i are similarly Y_i. Thus the y_i are trivial in homology, giving us that $\overline{G_{j,k}}$ must be generated by s and z, so equals $\mathbb{Z}\times\mathbb{Z}$.

On taking any $H \leq_f G$ we have a natural surjective homomorphism from the abelianisation H/H' of H to $H/H \cap G'$. So if we set H equal to $G_{j,k}$ we have $G' \leq G_{j,k}$, meaning that this surjective homomorphism is from $G_{j,k}/G'_{j,k}$ to $G_{j,k}/G' \leq_f G/G' = \mathbb{Z} \times \mathbb{Z}$. As $\mathbb{Z} \times \mathbb{Z}$ is Hopfian, we must have an isomorphism with $G'_{j,k} = G'$. Now if A is any finite abelian cover of G then there is $G_{j,k} \leq A$, but $G'_{j,k} \leq A' \leq G'$ so $A/A' = A/G' \leq_f G/G'$, giving $A/A' = \mathbb{Z} \times \mathbb{Z}$.

Finally if $N \trianglelefteq_f G$ is a soluble cover then on taking K with $N \trianglelefteq_f K \trianglelefteq_f G$ and $(G/N)/(K/N)$ abelian, we have $K/K' = \mathbb{Z} \times \mathbb{Z}$ with $K' = G'$. Now we can replace G with K and K with some subgroup L which is normal in K with K/L abelian. This allows us to repeat the argument above because we have $G' = K' \leq L \leq_f G$ so actually $L \trianglelefteq_f G$ with G/L abelian, thus again $L' = G'$ from above. We continue until we reach N, so in fact the soluble cover was abelian with $N' = G' \leq N$ and $N/N' \leq_f G/G' = \mathbb{Z} \times \mathbb{Z}$.

□

Therefore we require other covers in order to proceed. It was recently shown in [6] that mapping tori of free group endomorphisms are residually finite by using group schemes and other techniques in algebraic geometry. This can be combined with a short lemma in [33] to complete our result on largeness of mapping tori of injective endomorphisms of free groups. However in proving this, it will be enough just to have a non-abelian finite quotient. Although we will still use the result of [6] to ensure this happens, our requirement is much weaker than the full strength of residual finiteness and it is worthwhile investigating this condition in more detail. Therefore we continue with the last part of our proof of largeness seemingly unresolved, but this will be concluded in the next section.

We finish this section by looking at those mapping tori G of endomorphisms of free groups which contain a Baumslag-Solitar subgroup. Our result on largeness is not quite definitive because we need $\beta_1(G) \geq 2$ in order to apply our methods and we cannot show that G necessarily has a finite index subgroup with that property. However this is the only obstacle to largeness.

Theorem 4.9 *If $G = \langle \Gamma, t \rangle$ is a mapping torus of an endomorphism θ of the free group Γ of rank n which contains a Baumslag-Solitar subgroup $B(j, k)$ then either G is large or $G = B(1, k)$ or $\beta_1(H) = 1$ for all $H \leq_f G$.*

Proof. As usual we assume that θ is injective. By Proposition 4.4 we know that G can only contain Baumslag-Solitar subgroups of type $B(1, k)$ or

$B(k,k)$ for $k \neq 0$ and as we have already covered those which contain $B(1,1)$, we need only consider $B(1,k) \leq G$ for $k \neq \pm 1$. If there is some $H \leq_f G$ with $\beta_1(H) \geq 2$ then we can replace G by H because H is a mapping torus by Proposition 4.3 (iii) and $B(1,k) \cap H \leq_f B(1,k)$ so H contains a Baumslag-Solitar group of the same form (for instance use [43] which characterises these groups as the soluble groups of deficiency 1). Therefore by Proposition 4.4 (iii) we are looking at the case where we have a periodic conjugacy class of the form $w \in F_n \backslash \{1\}$ and $i > 0$ with $\theta^i(w)$ conjugate to w^d for some $d \neq \pm 1$. Just as in the $\mathbb{Z} \times \mathbb{Z}$ case, we drop down to a finite index subgroup and assume that $\theta(w) = w^d$. Now we follow the proof of Theorem 4.5 to get $F \leq_f \Gamma$ with $\langle w \rangle$ a free factor of F, observing that $w \in \theta^{-1}(F)$ so that we keep w as we pull back F. Note that we can assume w is not a proper power by Proposition 4.4 (ii), so we can also preserve w in a free basis each time because $w^d \in \theta\theta^{-1}(F)$ and if $w^c \in \theta\theta^{-1}(F)$ for $0 < |c| < |d|$ then the element $u \in \theta^{-1}(F)$ with $\theta(u) = w^c$ cannot be a power of w but $\theta(u^d) = \theta(w^c)$, thus contradicting injectiveness. Thus w^d can be extended to a free basis for $\theta\theta^{-1}(F)$ by [35] Proposition I.3.7 and thus w will be in the corresponding basis for $\theta^{-1}(F)$.

This gives an equivalent version of Corollary 4.6 where we have a deficiency 1 presentation with a relation $sxSX^d$. Thus on taking a surjective homomorphism χ to $\mathbb{Z}$ (which must send x to 0) we have as in Theorem 3.2 a top row with only one non-zero entry which is $t^{\chi(s)} - d \in \mathbb{Z}[t^{\pm 1}]$. If $\beta_1(G) = 1$ then the only available χ will send s to ± 1 but if not then we can find χ with $\chi(s) = 0$. Now take a prime p dividing $1 - d$ (and if d is rude enough to be 2 then take the double cover of G with relation $sxSX^4$). We then obtain largeness from Howie's criterion in Theorem 2.3 but with $\mathbb{Z}_p$ as our field.

□

Although we do not have a proof that a mapping torus of a free group endomorphism containing a Baumslag-Solitar subgroup of infinite index has a finite index subgroup with first Betti number at least two, the statement of Theorem 4.9 is still useful in a practical sense because if we are presented with a particular group G of this form that we would like to prove is large, we can enter the presentation into a computer and ask for the abelianisation of its low index subgroups. As soon as we see one with first Betti number at least two, we can conclude largeness.

Example 4.10

The group $G = \langle a, t | t^2 a t^{-2} = a^2 \rangle$ was shown in [18] to be a mapping torus of a free group endomorphism (just put $b = tat^{-1}$) which is 1-related and residually finite but not linear. It clearly contains $B(1,2)$ so we can conclude by Theorem 4.9 that it is a large 1-related residually finite non-linear group by getting Magma or GAP to tell us that it has a subgroup of index six with abelianisation $\mathbb{Z} \times \mathbb{Z}$.

We can even say something if G is a mapping torus of an injective endomorphism of an infinitely generated free group in the case when G is finitely generated, thanks to the power of [23] Theorem 1.2 which proves that G has a presentation of the form

$$\langle t, a_1, \ldots, a_k, b_1, \ldots, b_l | t a_1 t^{-1} = w_1, \ldots, t a_k t^{-1} = w_k \rangle$$

for $w_1, \ldots, w_k$ words in the a_i and the b_j. Thus either G has deficiency at least two and so is large, or $l = 0$ in which case G is also a mapping torus of a finitely generated free group endomorphism and so the results of this section apply.

5 Residually useless groups

Recall that a group G is residually finite if the intersection S over all the finite index subgroups $F \leq_f G$ is the trivial group I. Although this works perfectly well as a general definition, it is most useful when G is finitely generated and that will be our assumption in this section. Our motivation for the next definition is to ask: how badly can a group fail to be residually finite and what is the worst possible case? The first answer that would come to mind is when G $(\neq I)$ has no proper finite index subgroups at all, but we have been dealing with groups possessing positive first Betti number and hence infinitely many subgroups of finite index. By noting that elements outside the commutator subgroup G' cannot be in S, we obtain our condition.

Definition 5.1 *We say that the finitely generated group G is* **residually useless** *if*

$$G' = \bigcap_{F \leq_f G} F$$

but G is non-abelian.

Note that by excluding G being abelian, we have that G residually finite implies G is not residually useless which is of course what we want. Also G has no proper finite index subgroups if and only if $G = G'$ and G is residually useless or I. The definition has many equivalent forms, some of which are close to being mere rewordings, but the general idea is that a residually useless group cannot be distinguished from its abelianisation if one only uses standard information about its finite index subgroups.

Proposition 5.2 *Let G be finitely generated and non-abelian with commutator subgroup G', abelianisation $\overline{G} = G/G'$ and let S be the intersection of the finite index subgroups of G. The following are equivalent:*
(i) G is residually useless.
(ii) G/S is abelian.
(iii) G has no non-abelian finite quotient.
(iv) If $a_n(G)$ denotes the number of finite index subgroups of G having index n then $a_n(G) = a_n(\overline{G})$ for all n.
(v) For all $F \leq_f G$ we have $F' = G'$.
(vi) For all $F \leq_f G$ we have $F \cap G' = G'$.
(vii) For all $F \leq_f G$ we have $F' = F \cap G'$.

Proof. The equivalence of (i) with (ii) is immediate on seeing that we always have $S \leq G'$ if G is finitely generated, and likewise with (iii) on dropping down to a finite index normal subgroup. As for (iv), this is just using the index preserving correspondence between the subgroups of $\overline{G}$ and the subgroups of G containing G'.

As for the rest, we have that $F' \leq F \cap G' \leq G'$ whenever F is a subgroup of G. If (i) holds for G and F is a finite index subgroup then the intersection R of the finite index subgroups of F contains S (which is G') but is inside F' so F' and G' are equal, giving (v). This immediately implies (vi) and (vii) so we just require that they in turn imply (i). This is obvious for (vi) but not for (vii) until we spot [33] Lemma 3.2. If (i) fails then take $F \leq_f G$ and $g \in G'$ but $g \notin F$. Dropping down to $N \leq F$ with $N \trianglelefteq_f G$, we have $H = N\langle g\rangle \leq_f G$ and $H/N \cong \langle g\rangle/(N \cap \langle g\rangle)$. Thus $g \notin H'$ because by being outside N it survives under a homomorphism from H to an abelian group. But g is certainly in $H \cap G'$.

□

The importance of condition (vii) holding for G is that we fail to pick up extra abelianisation in finite covers $F \leq_f G$ since F/F' is just $F/F \cap G' \cong$

$FG'/G' \leq_f G/G'$. In particular $\beta_1(F) = \beta_1(G)$ so G is not large.

Example 5.3

(i) The Thompson group T is residually useless. This group has a 2-generator 2-relator presentation with abelianisation $\mathbb{Z}\times\mathbb{Z}$ and its commutator subgroup T' has no proper finite index subgroups as T' is infinite and simple. Thus for $F \leq_f T$ we have $F \cap T' \leq_f T'$ giving $T' \leq F$.
(ii) As already mentioned, if G is infinite but has no proper finite index subgroups then G is residually useless. Moreover for any such G and any finitely generated abelian group A we have $\Gamma = G * A$ is residually useless because if $N \trianglelefteq_f \Gamma$ then $N \cap G \trianglelefteq_f G$ so $G \leq N$. This implies that the normal closure C of G is in N so Γ/N must be abelian as it is a finite quotient of $\Gamma/C \cong A$. This also works if A is residually useless.
(iii) A famous example that will do for G in (ii) is the Higman group H with 4 generators and 4 relators, so has deficiency 0. Thus $H * H$ is residually useless so it too has no proper finite index subgroups, given that it is infinite and equals its own commutator subgroup. On taking the free product of lots of copies of H and topping it off with a deficiency 1 abelian group, namely $\mathbb{Z}$ or $\mathbb{Z}\times\mathbb{Z}$, we have examples of deficiency 1 residually useless groups G which need arbitrarily many generators and which have $\beta_1(G) = 1$ or 2.
(iv) We can ask if there are 1-relator groups which are residually useless. The presentation must have 2 generators (to avoid large or cyclic cases) so a first attempt would be the Baumslag-Solitar groups $G = B(l,m)$ with l and m coprime (for if not then G is large by [22] Theorem 6). We have in [22] Example 3.2 a proof that $\beta_1(H) = 1$ for all $H \leq_f G$ so these are not large, but they are not residually useless either.

Proposition 5.4 *The Baumslag-Solitar groups are not residually useless.*

Proof. If $G = B(l,m)$ where by replacing generators with their inverses we can take $l + m \geq 0$ and $l > 0$ then we can see that they surject to dihedral groups because the presentation

$$\langle x, y | xy^l x^{-1} = y^m, x^2 = y^{l+m} = 1, xyx^{-1} = y^{-1}\rangle$$

is a homomorphic image but the Baumslag-Solitar relation is redundant, leaving us the dihedral group of order $2(l+m)$. This gives us a finite non-abelian image unless $l + m = 1$ or 2. In the latter case we must have $l \geq 2$

and $m \leq -1$ as $B(1,1)$ is abelian. Now if G is residually useless and $F \leq_f G$ then F is too by Proposition 5.2 (v) so we drop to the index 2 subgroup with exponent sum of x equal to 0 modulo 2 and on rewriting we have

$$\langle t, y, z | t y^l t^{-1} = z^m, z^l = y^m \rangle$$

where $t = x^2$ and $z = xyx^{-1}$. Now as l and m are coprime this surjects to

$$\langle t, y, z, u | t y^l t^{-1} = z^m, u^j = 1, y = u^l, z = u^m \rangle$$

where $j = l^2 + m^2 \geq 5$. But on adding $tut^{-1} = u^{-1}$ and $t^2 = 1$ to the presentation we see the first relation goes as before and we obtain our dihedral image.

□

However in [22] the very next example due to Edjvet and Howie is that of an HNN extension of $B(2,3)$ given by

$$G = \langle x, y, z | x y^2 x^{-1} = y^3, z y z^{-1} = x^{-1} \rangle$$

which is a 2-generator 1-relator group shown to have the property that if $H \leq_f G$ then $\overline{H} = \mathbb{Z}$. This implies that G is residually useless because otherwise by Proposition 5.2 (vii) we would have some H with a homomorphism from $\overline{H}$ to a finite index subgroup of $\overline{G} = \mathbb{Z}$ which is surjective but not injective. As $\mathbb{Z}$ is Hopfian we would have $\overline{H} \neq \mathbb{Z}$. In fact the first example of a 1-relator residually useless group dates back to a short paper [1] of G. Baumslag in 1969 entitled "A non-cyclic one-relator group all of whose finite quotients are cyclic" with the group in question being

$$\langle a, b | a = a^{-1} b^{-1} a^{-1} b a b^{-1} a b \rangle.$$

In terms of its wide application, the following is our main result on largeness of deficiency 1 groups.

Theorem 5.5 *If G has a deficiency 1 presentation $\langle F_n | R \rangle$ where one of the relators is a commutator in F_n then exactly one of these occurs:*
(i) $G = \mathbb{Z} \times \mathbb{Z}$.
(ii) G is residually useless with abelianisation $\mathbb{Z} \times \mathbb{Z}$.
(iii) G is large.
In particular if $\overline{G} \neq \mathbb{Z} \times \mathbb{Z}$ then G is large.

Proof. If our relator $r = uvUV$ for u, v words in the generators for F_n then we can regard r as the commutator of two generators simply by adding u and v to the generators and their definitions to the relators, noting that this does not change the deficiency. As we have $\beta_1(G) \geq 2$, we have largeness by Theorem 3.2 and Proposition 3.7 unless $\overline{G} = \mathbb{Z} \times \mathbb{Z}$ with u and v linearly independent in homology. If so then either we are in (i) or (ii), or by Proposition 5.2 (vii) we have $L \leq_f G$ with γ in $L \cap G'$ but not in L', thus granting extra homology but we need to take care that we retain the form of our special relator r. We do this by keeping track of what happens to r under Reidemeister-Schreier rewriting when we drop to a finite index subgroup. First we can assume without loss of generality that u and v generate the homology as we can drop to the appropriate finite abelian cover H. This works because whenever we rewrite with respect to a subgroup of G, any generators of G which are in the subgroup automatically become generators of this subgroup and any relator consisting just of these generators will survive in the presentation for H. Now if $\overline{H} \neq \mathbb{Z} \times \mathbb{Z}$ then we are done but if not then a failure to pick up extra abelianisation means as before that $H' = H \cap G'$ but $G' \leq H$, giving $H' = G'$.

Let k, l be the minimum positive integers such that $a = u^k$ and $b = v^l$ are in L. We set N to be the smallest abelian cover of H containing a and b, noting that N is an abelian cover not just of H but of G as well. In rewriting for N in H we take a Schreier transversal of the form $u^i v^j$ (for $0 \leq i < k, 0 \leq j < l$). Thus a and b will be amongst the generators for N with our relator $uvUV$ in H giving rise to $abAB$ in N. (It is probably easiest to see this in two stages by first dropping to the subgroup with exponent sum of u equal to 0 mod k and rewriting using the obvious transversal u^i, and then doing the same with v.) Thus we have $G' \leq N \trianglelefteq_f G$ with a deficiency 1 presentation including generators a, b and relator $abAB$.

Finally we go from N to the subgroup $L \cap N \leq_f G$ which on rewriting will keep a and b because they are generators in the presentation for N which also lie in $L \cap N$, and consequently $abAB$ remains too. Now our $\gamma \in L$ from before which is in $G' \backslash L'$ is also in N as $G' \leq N$. But we have a surjective but non-injective homomorphism from L/L' to $L/L \cap G' \cong \mathbb{Z} \times \mathbb{Z}$, thus by the Hopficity of finitely generated abelian groups L surjects to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_m$ with γ mapping onto the $\mathbb{Z}_m$ factor. But then we can restrict this surjection to the finite index subgroup $L \cap N$ which also contains γ so $L \cap N$ has the right presentation and the right homology to obtain largeness.

□

We can now finish off Section 4 properly and with the minimum of fuss.

Corollary 5.6 *If $G = \langle \Gamma, t\rangle$ is the mapping torus of an endomorphism θ of the free group Γ of rank n and $\mathbb{Z}\times\mathbb{Z} \leq G$ then $G = \langle x, y | xyx^{-1} = x^{\pm 1}\rangle$ or is large.*

Proof. By the comment at the beginning of Section 4, we can assume that θ is really an injective endomorphism of a free group F_m with $m \leq n$, putting us in the case of Corollary 4.6 which allows us to apply Theorem 5.5 to $H \leq_f G$. As $\mathbb{Z}\times\mathbb{Z} \leq G$, we do not have $m = 0$ and only the two groups above for $m = 1$. Otherwise G and hence H contain a non-abelian free group for $m \geq 2$ so H is not in case (i) of Theorem 5.5. By Proposition 4.3 (iii) H is an injective mapping torus and so the result [6] of Borisov and Sapir tells us that H is residually finite, not residually useless. Thus H and G are large.
□

We also have a strong result for largeness of 1-relator groups. Obviously if a 1-relator presentation has three or more generators then we have largeness, so we want 2-generator 1-relator large groups.

Corollary 5.7 *If $G = \langle a, b | uvUV\rangle$ where u and v are any elements of $F_2 = \langle a, b\rangle$ with $uvUV$ not equal to $abAB$, $baBA$ or their cyclic conjugates when reduced and cyclically reduced then G is large or is residually useless.*

Proof. It is well known that $G = \mathbb{Z}\times\mathbb{Z}$ if and only if the relator is of the above form (equivalently if and only if u, v form a free basis for F_2) so otherwise we are in Theorem 5.5 case (ii) or (iii).
□

This begs the question: are there groups of the above form which are residually useless? If not then we have a definitive result on largeness for 1-relator groups G whose relator is a commutator; moreover on being given any such relator it is clear from Corollary 5.7 how one would immediately work out whether the group is large or is $\mathbb{Z}\times\mathbb{Z}$. Note that G has a deficiency 1 presentation with $\beta_1(G) \geq 2$ and the only examples we know of residually useless groups with these properties are ones of the form $H * (\mathbb{Z}\times\mathbb{Z})$ where $H \neq I$ has deficiency zero and no proper finite index subgroups, as in Example 5.3 (iii), but these will need more than two generators. In fact all known examples of 2-generator 1-relator groups with the relator a commutator are even residually finite; this is Problem (OR8) in the problem list at [38]. It seems likely that they are all not residually useless and hence large (or $\mathbb{Z}\times\mathbb{Z}$). In fact it is easy to prove this for most cases.

Proposition 5.8 *If $G = \langle F_2 | uvUV \rangle$ then G can only be residually useless if $u, v \notin F_2'$ with the images of u and v linearly independent in the abelianisation $\mathbb{Z} \times \mathbb{Z}$ of F_2 and such that u is a free basis element for F_2 or $G_u = \langle F_2 | u \rangle$ is residually useless, along with the same condition for v.*

Proof. If the images of u and v do not generate the homology of F_2 up to finite index then we are done by Theorem 3.2. Now suppose that G_u is not residually useless or $\mathbb{Z}$ (the latter happening if and only if u is an element of a free basis for F_2) then as G surjects to G_u we see that a non-abelian finite image of G_u is also an image of G. Now swap u and v.

□

There are many other cases for which we can conclude that $G = \langle F_2 | uvUV \rangle$ is not residually useless and hence large. The powerful algorithm of K. S. Brown in [10] to determine whether a 2-generator 1-relator group is a mapping torus of an injective endomorphism of a finitely generated group (which must necessarily be free), along with [6] proving that such groups are residually finite, suggests that on being given a particular G we proceed as follows: first check that $uvUV$ is not conjugate to $abAB$ or its inverse (else $G = \mathbb{Z} \times \mathbb{Z}$) and that u and v are independent in homology (else G is large). Now see if G_u or G_v can be shown to be not residually useless and not abelian by checking the abelianisation of low index subgroups of each of them. If either succeeds then G is large. If not then draw out the words u, v and $uvUV$ whilst using Brown's algorithm to determine the BNS invariants of G_u, G_v and G. If any of these are non-empty then we have largeness of G (although if the BNS invariant of G_u is non-empty then we require $G_u \neq \mathbb{Z}$ but this is easily checked given that $\beta_1(G_u) = 1$, because if the BNS invariant consists of both points of S^0 and the Alexander polynomial is 1 then $G_u = \mathbb{Z}$, with the same for G_v). If all these fail then compute the abelianisation of low index subgroups of G and look for one which is not $\mathbb{Z} \times \mathbb{Z}$ in order to obtain largeness. Needless to say, we know of no G which fails all of these tests.

We finish with a case where we can use Theorem 5.5 in practice. In [20] the problem of when

$$G = \langle a, b | a^{k_1} b^{l_1} a^{k_2} b^{l_2} a^{k_3} b^{l_3} \rangle$$

for $k_1 k_2 k_3, l_1 l_2 l_3 \neq 0$ and $k_1 + k_2 + k_3 = l_1 + l_2 + l_3 = 0$ is large, namely the case where the relator is in the derived subgroup and has free product length 6. It was shown that G is large apart from possibly the cases of one relator

and two infinite families of relators which were

$$\begin{array}{rll} A^3T^2aTa^2t^3, & & \\ a^{k_1}TaTa^{k_3}t^2 & \text{for} & k_1+k_3=-1, \\ a^{k_1}Ta^2Ta^{k_3}t^2 & \text{for} & k_1+k_3=-2. \end{array}$$

Note that by [35] Proposition I.8.4 it is straightforward to tell whether an element r in a free group is a commutator as we must have $r = uvwUVW$ for some reduced words u, v, w where this expression involves no cancellation. Thus we can chop the word in half and check the finite number of possibilities for u, v, w.

Corollary 5.9 *The groups above are all large.*

Proof. Writing $[u, v]$ for $uvUV$, the second case is $[a^{k_1}Ta, T^2a]$ and the third is $[a^{k_1}Ta^2, T^2a^2]$, so Theorem 5.5 applies and we are not in case (i) as the words are cyclically reduced. By drawing out the relators on a 2 dimensional grid using Brown's algorithm, we conclude that the groups are free-by-$\mathbb{Z}$ and hence large. As for the first relator, this is not a commutator so we turn to the computer. We find an index 4 subgroup with abelianisation $\mathbb{Z}^5$ and on rewriting we are told that it has a presentation with the five generators $x = at$, $y = ta$, $z = t^2A^2$ and a^3T, a^2tA along with four relators including one which is $ZyxzYX$, a commutator.

□

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